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Mirror symmetry in $2+1$ and $1+1$ dimensions

Mina Aganagic and Kentaro Hori

Jefferson Physical Laboratory, Harvard University
Cambridge, MA 02138, U.S.A.
E-mail: mina@cartan.harvard.edu, hori@infeld.harvard.edu

Andreas Karch

Center for Theoretical Physics, Massachusetts Institute of Technology
Cambridge, MA 02139, U.S.A.
E-mail: karch@ctp.mit.edu

David Tong

Department of Physics, Columbia University
New York, NY 10027, U.S.A.
E-mail: tong@physics.columbia.edu

Abstract: We study the Coulomb-Higgs duality of $\mathcal{N}=2$ supersymmetric abelian Chern-Simons theories in $2+1$ dimensions, by compactifying dual pairs on a circle of radius R and comparing the resulting $\mathcal{N}=(2,2)$ theories in $1+1$ dimensions. Below the compactification scale, the theory on the Higgs branch reduces to the non-linear sigma model on a toric manifold. In the dual theory on the Coulomb branch, the Kaluza-Klein modes generate an infinite tower of contributions to the superpotential. After resummation, in the limit $R\to 0$ the superpotential becomes that of the Landau-Ginzburg model which is the two-dimensional mirror of the toric sigma model. We further examine the conjecture of all-scale three-dimensional mirror symmetry and observe that it is consistent with mirror symmetry in $1+1$ dimensions.

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Contents

1. Introduction

Many of the dualities of interacting quantum field theories exchange Noether charges and topological charges and can be considered as generalizations of the abelian duality of free field theories. However, while abelian dualities in various dimensions are related by dimensional reduction, the same is not true of interacting theories. One reason for this is that non-trivial dualities are usually an infra-red equivalence while compactification probes short-distance scales. In this paper, we present one example where non-trivial dualities are related by compactification. The dualities in question are the mirror symmetry of $(2,2)$ supersymmetric field theories in $1+1$ dimensions [1] and the Coulomb-Higgs duality of supersymmetric gauge theories in $2+1$ dimensions, which is also known as mirror symmetry [2].

Mirror symmetry of $(2,2)$ theories in $1+1$ dimensions can be considered as a generalization of scalar-scalar duality (T-duality). It exchanges the vector and axial U(1) R-symmetries, together with the symplectic geometric and the complex analytic aspects of the theories. A proof this duality for a class of theories was recently offered in [1]: one starts with a gauged linear sigma-model and performs T-duality on each charged chiral multiplet. The new, dual, variables that one obtains are naturally adapted for describing instantons which, in two dimensional gauge theories, are Nielsen-Olesen vortices. After accounting for such effects, the resulting mirror theory is a Landau-Ginzberg (LG) model with a Toda-type superpotential.

On the other hand, mirror symmetry of gauge theories in $2+1$ dimensions was originally proposed for theories with $\mathcal{N}=4$ supersymmetry [2], and can be thought of as a generalization of scalar-vector duality. It exchanges the two SU(2) R-symmetry groups (hence the name "mirror symmetry"), and correspondingly, the Coulomb and the Higgs branches. This duality is not as well understood as the two-dimensional mirror symmetry in the sense that no derivation has been given except via embeddings into string theory [3], which involves the subtle issue of decoupling extra degrees of freedom. Nevertheless there exists considerable field theoretic evidence. Moreover, there exists a tantalising similarity with the transformation used in [1]: mirror symmetry in three dimensions exchanges the bound states of electrons with Nielson-Olesen vortices [4]. One may therefore wonder whether there exists a deeper connection between the two dualities.

In order to make more quantitative contact with the two-dimensional theories, we must firstly flow to three-dimensional mirror pairs with $\mathcal{N}=2$ supersymmetry. A prescription for this was given in [5] by weakly gauging a diagonal combination of the R-symmetry currents. This induces mass splittings for the hypermultiplets which, in turn, leads to the dynamical generation of Chern-Simons couplings. For abelian theories, the resulting Chern-Simons mirror pairs were previously analyzed in [6] where it was shown that the Coulomb and Higgs branches do indeed coincide as toric varieties.

In this paper, we compactify the mirror pairs of [6, 5] on a circle of radius R and study the two-dimensional "continuum" limit $R \to 0$. We tune the parameters of one theory (on the Higgs branch) to ensure that it descends to a two-dimensional non-linear sigma model on a toric manifold. The corresponding dual theory (on the Coulomb branch) is then analyzed in the same limit, where it appears as a $1+1$ dimensional gauge theory with infinitely many Kaluza-Klein (KK) matter fields. Each of the charged KK modes generates a superpotential term on the Coulomb branch, computed exactly at the one-loop level, and we sum this infinite tower of terms. In the limit $R \to 0$, we obtain the superpotential for the LG mirror [1] of the toric sigma model.

Thus, we show that three-dimensional mirror symmetry descends, upon compactification, to two-dimensional mirror symmetry. This may come as something of a surprise since the former is originally regarded as an infra-red duality. There is, however, no surprise: the superpotential is a holomorphic quantity and is independent of the ratio of the compactification scale and the scale set by the three-dimensional gauge coupling. Equipped with an infra-red duality, one would therefore expect to be able to re-derive agreement between such holomorphic quantitites, while non-holomorphic objects, such as the Kahler potential, would be beyond our reach. However, there is a proposal [7] that, after a suitable modification of the theories, three-dimensional mirror symmetry holds beyond the infra-red limit. Given this proposal, we also attempt a comparison of the Kahler potential, and indeed find agreement with two-dimensional mirror symmetry [1, 8]. This observation can be regarded as support for the 3d duality at all length scales.

The rest of the paper is organized as follows. In section 2, we warm up by recalling some basic facts on compactification from 2+1 to 1+1 dimensions. Section 3 is the main part of this paper. We study the compactification of the Model A/Model B mirror pairs of [6, 5]. We tune the parameters so that Model A reduces to the non-linear sigma model on a toric manifold. For Model B, we compute the exact superpotential after compactification and show that in the limit $R \to 0$ it becomes the LG superpotential of the 2d mirror. In section 4, we further study the Kahler potential of the all-scale mirror pairs proposed in [7], which requires a modification of Model A. Upon compactification we obtain two-dimensional mirror symmetry between the sigma model on a sqashed toric manifold and the LG model with a finite Kahler potential, in agreement with two-dimensional mirror symmetry. In section 5, we discuss the BPS states of the three-dimensional mirror pairs and, in particular, the relationship to the compactification analysis. In section 6 we conclude and outline some directions for future research. We also include two appendices. In appendix A, we specify the regularization scheme we are using in this paper and resolve a subtle issue in the compactification analysis of section 3. In appendix B, we review the Coulomb branch analysis of [6] and strengthen the statement by proving a non-perturbative non-renormalization theorem of the Kahler class.

2. Compactification preliminaries

In this section we discuss general aspects of $\mathcal{N} = 2$ supersymmetric gauge theories in $2+1$ dimensions compactified on a circle S^1 of radius R. (Such compactifications were considered, for example, in [9, 10, 11].) We denote the space-time coordinates by x^0, x^1, x^2 where x^0, x^1 are the time and space coordinates of the infinite $1+1$ dimensions and x^2 is the coordinate of the circle S^1, with period $2\pi R$.

The relationship between the two and three-dimensional coupling constants is,

$$\frac{2\pi R}{e_{3\mathrm{d}}^2} = \frac{1}{e_{2\mathrm{d}}^2} \,. \tag{2.1}$$

In general the physics of the system depends on the dimensionless combination

$$\gamma = 2\pi R\, e_{3\mathrm{d}}^2 \,. \tag{2.2}$$

For $\gamma \gg 1$, the system flows first to infra-red three-dimensional physics, before flowing to two dimensions. In contrast, for $\gamma \ll 1$, the system is essentially two-dimensional by the time strong three-dimensional gauge interactions play a role.

From the point of view of two-dimensions, a three-dimensional field consists of a tower of infinitely many Kaluza-Klein fields, each of which is a mode of the Fourier expansion in the x^2 direction. The Kaluza-Klein fields of a three-dimensional chiral multiplet of real mass m are all two-dimensional chiral multiplets, with twisted masses given by $(m + in/R)$, for integer n. A three-dimensional abelian vector multiplet contains a single real scalar ϕ, a Dirac fermion, and a U(1) gauge field v_μ, together with an auxillary scalar D. Its Kaluza-Klein modes are a two-dimensional vector multiplet V, together with an infinite tower of massive Kaluza-Klein modes. The two-dimensional vector multiplet contains a complex scalar field which decomposes into $\sigma = \sigma_1 + i\sigma_2$ with

$$\begin{aligned} \sigma_1 &= \frac{1}{2\pi R}\int_{S_1} \phi \\ \sigma_2 &= \frac{1}{2\pi R}\int_{S_1} v \equiv \sigma_2 + \frac{1}{R} \,, \end{aligned} \tag{2.3}$$

where the periodicity of the Wilson line σ_2 arises from large gauge transformations. Other fields in V descend by obvious dimensional reduction from three-dimensions. They can be combined into a gauge invariant twisted chiral superfield $\Sigma = \overline{D}_+ D_- V$ whose lowest component is σ. The massive vector Kaluza-Klein modes are also twisted chiral multiplets whose scalar components are the non-constant Fourier modes of ϕ and the Wilson line. These multiplets have twisted masses in/R for integer n.

2.1 Fayet-Iliopoulos and Chern-Simons couplings

The three-dimensional FI parameter ζ and the two-dimensional FI parameter r are related by

$$r = 2\pi R \zeta \,. \tag{2.4}$$

In the two-dimensional theory, the FI coupling is given by a twisted superpotential

$$\widetilde{W}_{FI} = -r\Sigma = -2\pi R\zeta\Sigma \,. \tag{2.5}$$

Let us see how the CS coupling is described in two-dimensions. Its bosonic part is given by

$$CS = \frac{k}{4\pi} \int\limits_{\mathbb{R}^2\times S^1} \left(v \wedge \mathrm{d}v + 2\phi D\, \mathrm{d}^3x\right). \tag{2.6}$$

For constant modes the intergal becomes two-dimensional with the integrand given by $Rk\ \mathrm{Re}\{\sigma(D - iv_{01})\}$. The supersymmetric completion amounts to the twisted superpotential

$$\widetilde{W}_{CS} = -\pi Rk\Sigma^2 . \tag{2.7}$$

Alternatively, we may derive this entirely within superspace: the supersymmetric Chern-Simons interaction is given in terms of the linear superfield $G = \epsilon^{\alpha\beta}\overline{D}_\alpha D_\beta V$. The dimensional reduction of this superfield may be expressed in terms of a twisted superfield, $G = \Sigma + \overline{\Sigma}$. The Chern-Simons interaction now reads

$$CS_{\mathrm{SUSY}} \ = \ -\frac{k}{4\pi}\int_{\mathbb{R}^2\times S^1} \mathrm{d}^3x\mathrm{d}^4\theta\, GV \ = \ -\frac{2\pi Rk}{4\pi}\mathrm{Re}\int_{\mathbb{R}^2} \mathrm{d}^2x\mathrm{d}^2\widetilde{\theta}\ \Sigma^2 \tag{2.8}$$

in agreement with the twisted superpotential (2.7).

2.2 Abelian duality in $2 + 1$ and $1 + 1$ dimensions

In $2 + 1$ dimensions, a U(1) gauge field is dual to a compact scalar field. Let us see how this duality looks like when compactified on a circle. We consider here the simplest free bosonic theory. In the free theory, the massive Kaluza-Klein modes are not coupled to the zero mode and simply go away in the limit $R \to 0$.

The three-dimensional duality is between a scalar field φ of period 2π with the action

$$S_{3d} = \frac{1}{2\pi}\int \mathrm{d}^3x \frac{\lambda^2}{2}\left((\partial_0\varphi)^2 - (\partial_1\varphi)^2 - (\partial_2\varphi)^2\right), \tag{2.9}$$

and a U(1) gauge field v_μ with the action

$$\widehat{S}_{3d} = \frac{1}{2\pi}\int \mathrm{d}^3x \frac{1}{2\lambda^2}\left((v_{01})^2 + (v_{02})^2 - (v_{12})^2\right). \tag{2.10}$$

The gauge coupling of the dual theory is λ, and thus the γ parameter of the compactification is given by

$$\widehat{\gamma} = 2\pi R\lambda^2 . \tag{2.11}$$

The compactification of the scalar field theory reduces in the $R \to 0$ limit to the $1 + 1$ dimensional theory with the action

$$S_{2d} = \frac{1}{2\pi}\int \mathrm{d}^2x \frac{\widehat{\gamma}}{2}\left((\partial_0\varphi)^2 - (\partial_1\varphi)^2\right). \tag{2.12}$$

This is the action for a sigma model on a circle of radius $\sqrt{\widehat{\gamma}}$. For the dual gauge theory, we note that the Wilson line σ_2 (2.3) has a finite periodicity after the rescaling

$\vartheta := 2\pi R\sigma_2 \equiv \vartheta + 2\pi$. In terms of the rescaled variable, the compactified theory reduces in the limit $R \to 0$ as

$$\widehat{S}_{2d} = \frac{1}{2\pi}\int \mathrm{d}^2x \frac{1}{2\widehat{\gamma}}\left((\partial_0\vartheta)^2 - (\partial_1\vartheta)^2 + (2\pi R v_{01})^2\right). \tag{2.13}$$

In $1+1$ dimensions, a Maxwell field has no propagating modes, and also, it has no finite energy topological excitation in the limit $\sqrt{\widehat{\gamma}}/R \to \infty$. Thus, the system (2.13) describes the sigma model on a circle of radius $1/\sqrt{\widehat{\gamma}}$, which is T-dual to (2.12).

We have seen that the abelian duality in $2+1$ dimensions descends to the abelian duality in $1+1$ dimensions. In the language of the gauge theory, T-duality is between the dual photon φ and the Wilson line ϑ. In the following, we shall see an analogous phenomenon in interacting, supersymmetric theories. Unlike in the free theory however, the massive Kaluza-Klein modes play an important role.

3. Compactification of 3D mirrors

In this section, which is the main part of this paper, we consider compactification of a mirror pair of interacting $\mathcal{N} = 2$ gauge theories in $2+1$ dimensions. We will tune the parameters (as a function of the compactification scale $1/2\pi R$) so that one theory reduces to the $1+1$ dimensional non-linear sigma model on the Higgs branch. Then, we will see that the other theory on the Coulomb branch reduces to the Landau-Ginzburg model in $1+1$ dimensions, which is the 2d mirror [1] of the sigma model on the Higgs branch.

3.1 Aspects of the 3D models

We consider mirror pairs of $\mathcal{N} = 2$ abelian Chern-Simons gauge theories found in [6].

Model A: $\mathrm{U}(1)^k$ gauge group with N chiral multiplets, Φ_i, $(i = 1, \ldots, N)$ of charge Q_i^a under the a^{th} U(1) factor $(a = 1, \ldots, k)$. The Chern-Simons couplings are given by $k^{ab} = \frac{1}{2}\sum_{i=1}^N Q_i^a Q_i^b$. The theory is parametrized by the gauge coupling constants e_a, the FI parameters ζ^a and the real masses m_i for Φ_i.

Model B: $\mathrm{U}(1)^{N-k}$ gauge group with N chiral multiplets $\widehat{\Phi}_i$, $(i = 1, \ldots, N)$ of charge $\widehat{Q}_i^p$ under the p^{th} U(1) factor $(p = 1, \ldots, N-k)$. The Chern-Simons couplings are given by $\widehat{k}^{pq} = -\frac{1}{2}\sum_{i=1}^N \widehat{Q}_i^p \widehat{Q}_i^q$. The theory is parametrized by the gauge coupling constants $\widehat{e}_p$, the FI parameters $\widehat{\zeta}^p$ and the real masses $\widehat{m}_i$ for $\widehat{\Phi}_i$.

The two sets of charges obey

$$\sum_{i=1}^N Q_i^a \widehat{Q}_i^p = 0, \quad \forall\, a \quad \text{and} \quad \forall\, p. \tag{3.1}$$

The mass and the FI parameters are related by the mirror map

$$\zeta^a - \frac{1}{2}\sum_{i=1}^{N} Q_i^a m_i = \sum_{i=1}^{N} Q_i^a \widehat{m}_i \,, \tag{3.2}$$

$$-\sum_{i=1}^{N} \widehat{Q}_i^p m_i = \widehat{\zeta}^p + \frac{1}{2}\sum_{i=1}^{N} \widehat{Q}_i^p \widehat{m}_i \,. \tag{3.3}$$

Of the $(N+k)$ mass and FI parameters describing Model A, only N are physical. This is because a shift of the scalar fields in the vector multiplets may be compensated by a shift of the parameters

$$m_i \to m_i + \sum_{b=1}^{k} Q_i^b c_b \,, \qquad \zeta^a \to \zeta^a + \sum_{b=1}^{k} k^{ab} c_b \,. \tag{3.4}$$

A similar remark applies to Model B where, once again, only N out of the $(2N-k)$ mass and FI parameters are physical. The mirror map (3.3) is the relation between these physical parameters.

The theories are not finite and the FI parameters have to be renormalized. Therefore, the mirror map (3.3) between FI and mass parameters depends on the regularization scheme. In fact, the mirror pairs [12] of finite $\mathcal{N}=4$ theories can serve as cut-off theories, and a specific regularization scheme is chosen to find the mirror map (3.3). The details are recorded in appendix A.

The Higgs branch of Model A corresponds to the Coulomb branch of Model B. While the Higgs branch is determined at the classical level by a symplectic quotient, the structure of the Coulomb branch arises due to certain quantum effects particular to $2+1$ dimensions. Specifically, there exist massless photons if, after integrating out the massive chiral multiplets, the effective Chern-Simons coefficients vanish. This defines the base of the Coulomb branch. The dual photons then provide the torus fibration. At the full quantum level, the Higgs branch of Model A and the Coulomb branch of Model B are exactly the same toric variety [6] with the same Kähler class, as shown in appendix B.

Unlike in theories with $\mathcal{N}=4$ supersymmetry, integrating out the chiral multiplets leads to a finite renormalization of the FI parameter. Thus, while the full Higgs branch exists only for vanishing masses, the Coulomb branch exists for vanishing *effective* FI parameter. In the region in which the Chern-Simons coupling vanishes, the effective FI parameter is the following physical combination

$$\begin{aligned} \zeta^a_{\text{eff}} &= \zeta^a - \frac{1}{2}\sum_{i=1}^{N} Q_i^a m_i \\ \widehat{\zeta}^p_{\text{eff}} &= \widehat{\zeta}^p + \frac{1}{2}\sum_{i=1}^{N} \widehat{Q}_i^p \widehat{m}_i \,, \end{aligned} \tag{3.5}$$

where the $\pm$ signs may be traced back to the $\pm$ signs of the Chern-Simons couplings. Notice that it is these effective FI parameters which appear in the mirror map (3.3).

In the past, little attention has been paid to the normalization of the mirror map. However, this will prove to be crucial for our story. We use conventions in which the auxiliary D-fields appear in the action as $\frac{1}{2\pi}\int d^3x(\frac{1}{2e^2}D^2 - \zeta D)$, as in [1]. Then, (3.3) is the correct one. This can be most easily seen by comparing the masses of the BPS vortices at special points of the Higgs branch with the masses of BPS electrons at special points of the Coulomb branch and recalling that these states are exchanged under mirror symmetry [4]. The check, with the sign, can also be made using the manipulation in [7].

3.2 Compactification of Model A on the Higgs branch

We compactify Model A on the circle of radius R so that at energies below $1/R$ we obtain the supersymmetric gauge theory in $1+1$ dimensions with the same gauge group and field content. We will focus on the Higgs branch of the model.

The FI parameters in two-dimensional gauge theories are renormalized as

$$r^a(\mu') = r^a(\mu) + \sum_{i=1}^{N} Q_i^a \log\left(\frac{\mu'}{\mu}\right). \tag{3.6}$$

This applies, in the present case, only at energies below the compactification scale

$$\Lambda_{\mathrm{UV}} = 1/2\pi R. \tag{3.7}$$

Thus, in order to have the renormalized two-dimensional theory, the three-dimensional FI parameters must depend on the radius R as

$$\zeta^a = \frac{1}{2\pi R} r^a(\Lambda_{\mathrm{UV}}) = \frac{1}{2\pi R}\left(\sum_{i=1}^{N} Q_i^a \log\left(\frac{1}{2\pi R\Lambda}\right) + r^a\right), \tag{3.8}$$

where the scale Λ and parameters r^a are taken fixed in the continuum limit $1/R \to \infty$ of the two-dimensional theory. If $\sum_{i=1}^{N} Q_i^a \neq 0$ for some a, Λ is a physical RG invariant scale parameter of the theory which replaces one combination of r^a (for $k=1$ it is standard to take $r=0$). If $\sum_{i=1}^{N} Q_i^a = 0$ for all a, all r^a's are physical parameters of the theory.[1]

We now set the FI parameters to be sufficiently large and all the real masses to zero. This ensures that, at low enough energies, the theory is the non-linear sigma model on the Higgs branch in $1+1$ dimensions. The Higgs branch is a toric manifold

[1]We could dispense with these remarks if we wrote (3.8) as $\zeta^a = \frac{1}{2\pi R}(\sum_{i=1}^{N} Q_i^a \log(1/2\pi R\mu) + r^a(\mu))$ as usual. Instead, we express it in terms of the RG invariant scale Λ (and parameters r^a) since the running 2d coupling may be confusing in what follows.

obtained as the symplectic quotient of $\mathrm{U}(1)^k$ acting on $\mathbb{C}^N$ with charge Q_i^a. Namely, the vacuum manifold

$$\sum_{i=1}^{N} Q_i^a |\Phi_i|^2 = \zeta^a \tag{3.9}$$

modded out by the $\mathrm{U}(1)^k$ gauge group action. In the above expression, Φ_i is the scalar component of the three-dimensional chiral multiplet Φ_i. In particular, these scalars have classical vacuum expectation values of order $\sqrt{\zeta}$ or

$$|\Phi|^2 \sim \zeta \sim \frac{1}{R}. \tag{3.10}$$

In the above discussion we have implicitly assumed $\gamma = e_{3d}^2 R$ to be small. However, this is not mandatory. We may take $\gamma \gg 1$, in which case the theory reduces to the non-linear sigma model above the compactification scale $\Lambda_{\mathrm{UV}} = 1/2\pi R$. As we flow to energies below Λ_{UV} the metric starts to run. The FI parameters play the role of the Kahler class parameters which are known to be renormalized only at the one loop level [13] (see also [14] for a simple derivation). This leads once again to the same R dependence as (3.8), for ζ^a.

3.3 Compactification of Model B on the Coulomb Branch

We turn now to the compactification of Model B on the Coulomb branch. We consider the limit $\widehat{\gamma} := \widehat{e}_{3d}^2 R \ll 1$ such that the theory flows to $1+1$ dimensional gauge theory before $2+1$ dimensional gauge interactions become important. Thus, we will consider this theory as a gauge theory in $1+1$ dimensions. We will see that the Kaluza-Klien modes, considered as infinitely many charged and neutral matter fields, play an important role.

Firstly we determine the parameters of Model B using the mirror map (3.3). The vanishing of the real masses of Model A requires us to set the effective FI parameters to zero, ensuring that Model B has a Coulomb branch,

$$\widehat{\zeta}^p = -\frac{1}{2}\sum_{i=1}^{N} \widehat{Q}_i^p \widehat{m}_i\,. \tag{3.11}$$

From (3.8), we require the real masses of Model B to depend on the radius R as

$$\widehat{m}_i = \frac{1}{2\pi R}\left(\log\left(\frac{1}{2\pi R\Lambda}\right) + r_i\right), \tag{3.12}$$

where r_i solves the equation $\sum_{i=1}^{N} Q_i^a r_i = r^a$.

We next discuss the appropriate variables to describe the Coulomb branch of Model B. Recall that the scalar components, $\hat{\phi}$ of the Model B vector multiplets are related to the scalar components Φ of the Model A chiral multiplets as [2, 15]

$$\widehat{\phi} \sim |\Phi|^2\,. \tag{3.13}$$

Comparing with (3.10) we should rescale the Coulomb branch variables by $1/R$. Together with the supersymmetric partner, we rescale the superfield strength $\widehat{\Sigma}_p$ as

$$\widehat{\Sigma}_p = \frac{\Theta_p}{2\pi R}\,. \tag{3.14}$$

This is the same scaling we performed in the free theory of section 2.2. As in that case, the periodicity of the imaginary part of $\widehat{\Sigma}_p$ given in (2.3) reveals

$$\operatorname{Im}\Theta_p \equiv \operatorname{Im}\Theta_p + 2\pi\,. \tag{3.15}$$

With this choice of the fields, the tree level twisted superpotential coming from the CS and the FI terms is expressed as

$$\begin{aligned}\widetilde{W}_{CS} + \widetilde{W}_{FI} &= -\pi R \sum_{p,q} \widehat{k}^{pq}\widehat{\Sigma}_p\widehat{\Sigma}_q - 2\pi R \sum_p \widehat{\zeta}^p\widehat{\Sigma}_p \\ &= \frac{1}{8\pi R}\sum_{i,p,q}\widehat{Q}_i^p\widehat{Q}_i^q\Theta_p\Theta_q + \frac{1}{2}\sum_{i,p}\widehat{Q}_i^p\widehat{m}_i\Theta_p\,,\end{aligned} \tag{3.16}$$

where we have used $\widehat{k}^{pq} = -(1/2)\sum_{i=1}^N \widehat{Q}_i^p\widehat{Q}_i^q$ and (3.11).

We consider the theory at energy E much smaller than the compactification scale $1/R$. Naively one may expect the effect of all the massive KK modes to disappear in the extreme two-dimensional limit $ER \to 0$. However, as we now see, the rescaling of fields (3.14) and masses (3.12) ensures that this is not the case.

Let us first integrate out all the KK modes of the chiral multiplets. This can be done exactly by one-loop integral because the action is quadratic in these variables. The effect includes the generation of the standard $\Sigma\log(\Sigma)$ type twisted superpotential. Summing up all such modes we obtain

$$\varDelta\widetilde{W} = -\sum_{i=1}^N\sum_{n=-\infty}^{\infty}\left(\widehat{\Sigma}_i + \frac{in}{R}\right)\left(\log\left(\widehat{\Sigma}_i + \frac{in}{R}\right) - 1\right), \tag{3.17}$$

where

$$\widehat{\Sigma}_i = \sum_{p=1}^{N-k}\widehat{Q}_i^p\widehat{\Sigma}_p + \widehat{m}_i\,. \tag{3.18}$$

We still have to integrate out the non-zero modes of the three-dimensional vector multiplets and the high frequency modes of $\widehat{\Sigma}_p$. We claim that this does not induce any twisted superpotential. Note that the effective perturbation expansion parameters are given by $e_{2d}^2/\Sigma^2 \sim \widehat{\gamma}/\Theta^2$, which is small as long as $\widehat{\gamma} \ll 1$ (and Θ finite). Then any quantum correction depends on $\widehat{\gamma}$ but it cannot enter into the twisted superpotential since the gauge coupling e_{2d} is not a twisted chiral parameter. (There could also be tree-level corrections from elimination of heavy fields by the equations of motion. However, conservation of momentum in the compactified dimension ensures that the equations of motion are solved by setting these fields to zero.) Thus, (3.17) is the exact quantum correction to the twisted superpotential.

To perform the infinite sum, we firstly differentiate,

$$
\begin{aligned}
\frac{\partial \Delta \widetilde{W}}{\partial \widehat{\Sigma}_p} &= -\sum_{i,n} \widehat{Q}_i^p \log\left(\widehat{\Sigma}_i + \frac{in}{R}\right) \\
&= -\sum_{i=1}^{N} \widehat{Q}_i^p \log\left[2\pi R\widehat{\Sigma}_i \prod_{n=1}^{\infty}\left(1+\frac{R^2\widehat{\Sigma}_i^2}{n^2}\right)\right] - \sum_{i=1}^{N} \widehat{Q}_i^p \log\left[\frac{1}{2\pi R}\prod_{n=1}^{\infty}\left(\frac{n^2}{R^2}\right)\right].
\end{aligned}
\tag{3.19}
$$

The infinite product in the first term can be performed, resulting in the argument of the logarithm equal to $2\sinh(\pi R\widehat{\Sigma}_i)$. The second term appears divergent, but it should be considered to vanish in our specific regularization scheme that leads to (3.3). This is explained in detail in appendix A. (If we took another regularization, the mirror map (3.3) would contain a divergent term and this would cancel the second term of (3.19) in the end. See appendix A.) Thus, we obtain

$$
\frac{\partial \Delta \widetilde{W}}{\partial \widehat{\Sigma}_p} = -\sum_{i=1}^{N} \widehat{Q}_i^p \log(\mathrm{e}^{\pi R\widehat{\Sigma}_i} - \mathrm{e}^{-\pi R\widehat{\Sigma}_i}). \tag{3.20}
$$

In terms of the rescaled variable Θ_p defined by (3.14) we have

$$
\frac{\partial \Delta \widetilde{W}}{\partial \Theta_p} = -\frac{1}{2\pi R}\sum_{i=1}^{N} \widehat{Q}_i^p \log(\mathrm{e}^{\Theta_i/2} - \mathrm{e}^{-\Theta_i/2}), \tag{3.21}
$$

where

$$
\Theta_i = 2\pi R\widehat{\Sigma}_i = \sum_{q=1}^{N-k} \widehat{Q}_i^q \Theta_q + 2\pi R\widehat{m}_i . \tag{3.22}
$$

Now, we recall from (3.12) that $2\pi R\widehat{m}_i$ is logarithmically divergent as $R \to 0$. Thus, within the argument of the logarithm, $\mathrm{e}^{\Theta_i/2}$ dominates over $\mathrm{e}^{-\Theta_i/2}$ and we have the expansion

$$
\begin{aligned}
\frac{\partial \Delta \widetilde{W}}{\partial \Theta_p} &= -\frac{1}{2\pi R}\sum_{i=1}^{N} \widehat{Q}_i^p \left\{\frac{\Theta_i}{2} + \log(1-\mathrm{e}^{-\Theta_i})\right\} \\
&= -\frac{1}{2\pi R}\sum_{i=1}^{N} \widehat{Q}_i^p \left\{\frac{\Theta_i}{2} - \mathrm{e}^{-\Theta_i} + O((\mathrm{e}^{-\Theta_i})^2)\right\}.
\end{aligned}
\tag{3.23}
$$

Since the equation (3.12) shows

$$
\mathrm{e}^{-\Theta_i} = 2\pi R\Lambda\, \mathrm{e}^{-\sum_{q=1}^{N-k}\widehat{Q}_i^q\Theta_q - r_i}, \tag{3.24}
$$

the higher order terms $\frac{1}{2\pi R}O((\mathrm{e}^{-\Theta_i})^2)$ vanish in the limit $R \to 0$. Thus, in this limit we obtain

$$
\frac{\partial \Delta \widetilde{W}}{\partial \Theta_p} = \sum_{i=1}^{N} \widehat{Q}_i^p \left\{-\frac{1}{4\pi R}\sum_{q=1}^{N-k} \widehat{Q}_i^q\Theta_q - \frac{1}{2}\widehat{m}_i + \Lambda\, \mathrm{e}^{-\sum_{q=1}^{N-k}\widehat{Q}_i^q\Theta_q - r_i}\right\}. \tag{3.25}
$$

This can be integrated as

$$\Delta\widetilde{W} = -\frac{1}{8\pi R}\sum_{i,p,q}\widehat{Q}_i^p\widehat{Q}_i^q\Theta_p\Theta_q - \frac{1}{2}\sum_{i,p}\widehat{Q}_i^p\widehat{m}_i\Theta_p - \Lambda\sum_{i=1}^{N}\mathrm{e}^{-\sum_{q=1}^{N-k}\widehat{Q}_i^q\Theta_q - r_i}\,. \tag{3.26}$$

The terms quadratic and linear in Θ_p are cancelled by the tree level terms (3.16). The total twisted superpotential is therefore given by

$$\begin{aligned}\widetilde{W}_{\text{total}} &= \widetilde{W}_{CS} + \widetilde{W}_{FI} + \Delta\widetilde{W}\\ &= -\Lambda\sum_{i=1}^{N}\mathrm{e}^{-\sum_{p=1}^{N-k}\widehat{Q}_i^p\Theta_p - r_i}\,. \end{aligned}\tag{3.27}$$

This is precisely the LG superpotential obtained in [1] for the mirror of the non-linear sigma model with toric target space.

Inclusion of twisted mass

One may consider weakly gauging a flavour symmetry of Model A to introduce non-zero real masses m_i. In the infra-red, this deforms the sigma-model on the Higgs branch by introducing a potential proportional to the length squared of a holomorphic Killing vector associated to the flavour symmetry. It is a simple matter to track this deformation in Model B: It shifts $\widehat{m}_i$ and $\widehat{\zeta}^p$ by a finite amount. Change in $\mathrm{e}^{-2\pi R\widehat{m}_i}$ due to a finite shift of $\widehat{m}_i$ does not matter in the limit $R \to 0$. The expression (3.16) of $\widetilde{W}_{CS} + \widetilde{W}_{FI}$ in terms of Θ_p and new $\widehat{m}_i$ is shifted by $\sum_{i,p}\widehat{Q}_i^p m_i\Theta_p$. Thus, it fails to cancel the Θ-linear term from $\Delta\widetilde{W}$. The total twisted superpotential is therefore given by

$$\widetilde{W} = -\Lambda\sum_{i=1}^{N}\mathrm{e}^{-\sum_{p=1}^{N-k}\widehat{Q}_i^p\Theta_p - r_i} + \sum_{i,p}\widehat{Q}_i^p m_i\Theta_p \tag{3.28}$$

once again in agreement with [1].

4. Interpretation

In the previous section we examined the fate of three-dimensional field theories upon compactification on a circle. We found that the three-dimensional mirror pairs lead to two-dimensional mirror pairs. This may appear to be a consistent picture, but one should note the following. Three-dimensional mirror symmetry was originally conjectured to be an infra-red duality, applying only in the limit $e_{3d}, \widehat{e}_{3d} \to \infty$. Equivalence of the two compactified theories would hold, based on this conjecture, only when the compactification scale $1/R$ is much smaller than the scales e_{3d}^2 and $\widehat{e}_{3d}^2$, namely only for $\gamma \gg 1, \widehat{\gamma} \gg 1$. However, we have considered the regime $\widehat{\gamma} \ll 1$ in Model B compactification, but still we found an agreement with Model A compactification. In this section, we discuss the meaning of this observation.

4.1 No phase transition between $\widehat{\gamma} \gg 1$ and $\widehat{\gamma} \ll 1$?

Firstly, let us shift perspective slightly and repeat the calculations presented above, with a somewhat different philosophy. To this end, we consider only Model B on the Coulomb branch and examine two different descriptions of this theory: one in terms of the dual photon, the other in terms of the Wilson line. The former description is valid for $\widehat{\gamma} \gg 1$, where it results in a two-dimensional sigma-model with target space given by the three-dimensional Coulomb branch. As shown in appendix (based on [6]), this Coulomb branch is equivalent to the Higgs branch of Model A as a toric manifold and they have the same Kahler class. We stress that, at this juncture, we have made neither assumption nor conjecture about the properties of the three-dimensional theory. For the other description, in terms of the Wilson line, we may make progress in the regime $\widehat{\gamma} \ll 1$ where the tower of Kaluza-Klein modes may be integrated out as in the previous section, resulting in the LG-model (3.27). The parameter which interpolates between the two regimes, $\hat{\gamma}$, is a D-term deformation, and therefore the holomorphic data — the Kahler class of the sigma model and the twisted superpotential of the LG model — can be extrapolated to all values of $\hat{\gamma}$. Thus, under the assumption of the absence of the phase transition between the two regimes $\widehat{\gamma} \gg 1$ and $\widehat{\gamma} \ll 1$, one can rederive the two-dimensional mirror symmetry [1] from what we know for sure about three-dimensions (up to the treatment of the divergent term in (3.19) that does rely on some asumption on three-dimensional theories, to be discussed in appendix A). From this perspective, we see that two-dimensional mirror symmetry is indeed an exchange between the dual photon and Wilson line description, as in the free theory in section 2.2.

In other words, given the two-dimensional mirror symmetry established in [1], this observation suggests that there is no phase transition between $\widehat{\gamma} \gg 1$ and $\widehat{\gamma} \ll 1$. Semi-classical analysis of the two-dimensional theory implies [1] that the Kahler metric is the flat one in the LG mirror of the toric sigma model (that arizes in $\widehat{\gamma} \gg 1$ in the present set-up). If there is no phase transition, the Kahler metric for Θ_p in the regime $\widehat{\gamma} \ll 1$ should not be too different from the flat metric.

4.2 The D-term at $\widehat{\gamma} \ll 1$

In fact, in the regime $\widehat{\gamma} \ll 1$ one can analyze also the D-term since we have small expansion parameters $\widehat{\gamma}/\Theta^2$ and $\widehat{\gamma}$ as noted in the previous section. The integration over the matter (including all the KK modes) can be performed exactly at the one-loop level. This leads to the following effective metric on the Coulomb branch

$$\mathrm{d}\widehat{s}^2 = \sum_{p,q} \left(\frac{\delta_{p,q}}{\widehat{e}_p^2} + \frac{1}{2} \sum_{i,n} \frac{\widehat{Q}_i^p \widehat{Q}_i^q}{|\widehat{\sigma}_i + in/R|^2} \right) \mathrm{d}\overline{\widehat{\sigma}}_p \mathrm{d}\widehat{\sigma}_q \,, \tag{4.1}$$

where $\widehat{\sigma}_i = \widehat{Q}_i^p \widehat{\sigma}_p + \widehat{m}_i$. The matter integral also induces other interactions of vector multiplet fields including Kaluza-Klein modes $\widehat{\Sigma}(n)$. Each of them is a power of

$\widehat{\Sigma}(n)$'s with some derivatives, divided by some power of $|\widehat{\sigma}_i|$'s (of at least second order). Thus, (4.1) is the leading correction in the $\widehat{\gamma}/\Theta^2$ and $\widehat{\gamma}$ expansion, even after integrating out the vector multiplet KK modes and higher frequency modes of $\widehat{\Sigma}$.

The infinite sum in (4.1) can be perfomed by using the formula

$$\sum_{n=-\infty}^{\infty} \frac{1}{x^2+(a+2\pi n)^2} = \frac{\sinh x}{2x(\cosh x - \cos a)} \tag{4.2}$$

which is $\sim 1/2x$ for large x. Noting the rescaling $\widehat{\sigma}_p = \Theta_p/2\pi R$ (3.14) and the R dependence (3.12) of $\widehat{m}_i$, we find that the Kahler metric behaves as

$$\mathrm{d}\widehat{s}^2 = \sum_{p,q}\left(\frac{\delta_{p,q}}{\widehat{\gamma}_p} + \frac{1}{4}\sum_{i=1}^{N}\frac{\widehat{Q}_i^p\widehat{Q}_i^q}{(2\pi R\widehat{m}_i + \mathrm{Re}\widehat{Q}_i^r\Theta_r)}\right)\mathrm{d}\overline{\Theta}_p\mathrm{d}\Theta_q\,. \tag{4.3}$$

If we introduce the notation, $Y_i = \widehat{Q}_i^p\Theta_p + r_i$ which obeys $\sum_{i=1}^N Q_i^a Y_i = r^a$ and appears in the superpotential as $\widetilde{W} = -\Lambda\sum_{i=1}^N \mathrm{e}^{-Y_i}$, the metric is written as

$$\mathrm{d}\widehat{s}^2 = \sum_{p=1}^{N-k}\frac{1}{\widehat{\gamma}_p}|\mathrm{d}\Theta_p|^2 + \frac{1}{4}\sum_{i=1}^{N}\frac{|\mathrm{d}y_i|^2}{\log(\Lambda_{\mathrm{UV}}/\Lambda)+\mathrm{Re}\,y_i}\,. \tag{4.4}$$

We see that it approaches the flat metric $\sum_{p=1}^{N-k}|\mathrm{d}\Theta_p|^2/\widehat{\gamma}_p$ in the two-dimensional continuum limit $\Lambda_{\mathrm{UV}}/\Lambda\to\infty$.

If we were allowed to take the limit $\Lambda_{\mathrm{UV}}/\Lambda\to\infty$ first, then the Kahler potential would stay flat even in the regime $\widehat{\gamma}\gg 1$ (with a vanishingly small coefficient at $\widehat{\gamma}\to\infty$ as in [1]). In the next subsection, instead of discussing whether this exchange of limits is valid, we turn to the mirror of the Model B compactification with $\widehat{\gamma}\ll 1$.

4.3 3d Mirror Symmetry beyond IR duality

As mentioned before, three-dimensional mirror symmetry is originally conjectured as an infra-red duality. However, there is a proposal that in fact it extends to all energy scales, with a mild modification of the field contents and the interaction [7]. Let us introduce

Model A′: $\mathrm{U}(1)^{2N-k} = \prod_{a=1}^k \mathrm{U}(1)_a \times \prod_{p'=1}^{N-k}\mathrm{U}(1)'_{p'}\times\prod_{\widetilde{p}=1}^{N-k}\widetilde{\mathrm{U}}(1)_{\widetilde{p}}$ gauge theory with N chiral multiplets Φ_i of charge Q_i^a and $R_{ip'}$ under $\mathrm{U}(1)_a$ and $\mathrm{U}(1)'_{p'}$ but neutral under $\widetilde{\mathrm{U}}(1)_{\widetilde{p}}$. Non-zero Chern-Simons couplings are $k^{ab} = \frac{1}{2}\sum_{i=1}^N Q_i^aQ_i^b$, $k_{p'q'} = \frac{1}{2}\sum_{i=1}^N R_{ip'}R_{iq'}$, $k_{p'}^a = \sum_{i=1}^N Q_i^a R_{ip'}$ and $k_{p'}^{\widetilde{q}} = \delta_{p'}^{\widetilde{q}}$. The theory is parametrized by the gauge coupling constants $e_a, e_{p'}, \widetilde{e}_{\widetilde{p}}$, and N combinations of the FI parameters ζ^a, $\zeta'_{p'}$ and the real masses m_i (the FI parameters of $\widetilde{\mathrm{U}}(1)_{\widetilde{p}}$ can always be set equal to zero by a field redefinition).

Here we define $R_{ip'}$ as solutions to

$$\sum_{i=1}^{N} \widehat{Q}_i^p R_{iq'} = \delta^p_{q'} \,, \tag{4.5}$$

(the ambiguity of shift by Q_i^a actually does not matter). Model A$'$ may be derived by the argument of [5], starting with the all scale $\mathcal{N}=4$ mirror symmetry proposal of [7]. According to this proposal, Model A$'$ in the limit $e_a, e'_{p'} \to \infty$ is exactly dual to Model B, provided $\widetilde{e}_p = \widehat{e}_p$ and the FI and masses are related by (3.3) with $\zeta'_{p'} - \frac{1}{2}\sum_{i=1}^{N} R_{ip'} m_i = \sum_{i=1}^{N} R_{ip'} \widehat{m}_i$.

The action is quadratic in $\widetilde{V}_{\widetilde{q}}$ and one may dualize it to a periodic chiral superfield $P_{\widetilde{q}}$. Then, the Higgs branch is of $N + (N-k) - [k + (N-k)] = N - k$ dimensions, as in the case of Model A. In fact, it is identical to the Higgs branch of Model A as a toric manifold with a Kahler class. The difference is that now the torus fibres have a constant size deep in the interior of the base of the fibration. For example, let us consider the case of $N=2$, $k=1$ with $Q_i = 1$, $Q'_1 = 1/2, Q'_2 = -1/2$. Then, the metric of the Higgs branch is given by

$$\mathrm{d}s^2 = \left(\frac{1}{\widetilde{e}^2} + \frac{\zeta}{\zeta^2 - \rho^2}\right) \mathrm{d}\rho^2 + \left(\frac{1}{\widetilde{e}^2} + \frac{\zeta}{\zeta^2 - \rho^2}\right)^{-1} \mathrm{d}\varphi^2 \,, \tag{4.6}$$

where ρ is the coordinate of the base $[-\zeta, \zeta]$ and φ is the coordinate of period 2π of the torus fibre. We see that deep in the interior $-\zeta \ll \rho \ll \zeta$ of the base, the torus fibre has a constant radius. If ζ is much larger than $\widetilde{e}^2$, the radius is approximately $\widetilde{e}$.

After compactification, we obtain a non-linear sigma model on this squashed toric manifold (which we shall call squashed toric sigma model). We give R dependence to $2\pi R\zeta$ as (3.8), and it is much larger than $\widehat{\gamma} = 2\pi R\widetilde{e}^2$ which is held fixed. Thus, the squared radius of the torus fibre is $\widehat{\gamma}_p = 2\pi R\widetilde{e}_p^2$ at high enough energies in the two-dimensional theory. Since large and small $\widehat{\gamma}$ are different simply by the squashing factor, it is hard to imagine that there is a phase transition between $\widehat{\gamma} \gg 1$ and $\widehat{\gamma} \ll 1$.

The squashed toric sigma model is in the class of theories studied recently in [8] in detail. As in [8], one can apply the argument of mirror symmetry in [1]. The dual of Φ_i and P_q are twisted chiral superfields Y_i and $\widetilde{Y}_q$ of period $2\pi i$ ($\widetilde{Y}_q$ is in fact the fieldstrength $2\pi R\widetilde{\Sigma}_q$ of $\widetilde{\mathrm{U}}(1)_q$). The twisted superpotential is

$$\widetilde{W} = \sum_{a=1}^{k} \Sigma_a (Q_i^a Y_i - r^a) + \sum_{q=1}^{N-k} \Sigma'^q (R_{iq} Y_i - \widetilde{Y}_q) + \sum_{i=1}^{N} \mathrm{e}^{-Y_i} \,. \tag{4.7}$$

The semi-classical Kahler metric is given by

$$\mathrm{d}s^2 = \sum_{q=1}^{N-k} \frac{1}{\widehat{\gamma}_q} |\mathrm{d}\widetilde{y}_q|^2 + \frac{1}{2} \sum_{i=1}^{N} \frac{|\mathrm{d}y_i|^2}{\log(\Lambda_{\mathrm{UV}}/\Lambda) + \mathrm{Re}\, y_i} \,. \tag{4.8}$$

After integrating out Σ_a and Σ'^q, we obtain the constraints $\sum_{i=1}^N Q_i^a Y_i = r^a$, together with $\sum_{i=1}^N R_{iq} Y_i = \widetilde{Y}_q$, which are solved by $Y_i = \sum_{q=1}^{N-k} \widehat{\widetilde{Q}}{}_i^p \widetilde{Y}_q + r_i$. (It has been conjectured by Fendley and Intriligator that the supersymmetric squashed S^2 sigma model, the supersymmetric version of the sausage model of [16], is mirror to sine-Gordon model of finite Kahler potential.)

We see that the compactification of Model B is essentially the same as this two-dimensional mirror of the squashed toric sigma model. It has the same superpotential $\sum_{i=1}^N \mathrm{e}^{-Y_i}$ and the behaviour of the Kahler metric (4.4) is similar to the semi-classical metric (4.8) of the 2d mirror.[2] Thus, the all scale mirror symmetry conjecture [7] is consistent with the $1+1$ dimensional mirror symmetry that is already established.

5. Vortex-electron exchange

In [4] it was argued that mirror symmetry of three-dimensional gauge theories can be interpreted as an exchange of the electron and vortex descriptions (see also [7]). More precisely, mirror symmetry exchanges Nielson-Oleson vortices on the Higgs branch with bound states of logarithmically confined electrons on the Coulomb branch. In this section we will elaborate on this interpretation in our models, and study its relation to the compactification analysis.

As usual, the mass of a particle is bounded from below by the central charge of the supersymmetry algebra. $\mathcal{N}=2$ supersymmetry algebra contains one real central charge which is a linear combination of conserved charges. In Model A, on the Higgs branch, there are k topological charges — the vortex numbers. Note that there are no Noether charges since the global symmetry $\mathrm{U}(1)^{N-k}$ is generically spontaneously broken. In contrast, in Model B on the Coulomb branch, the global symmetry $\mathrm{U}(1)^k$ is unbroken and there are k Noether charges. Thus, in both theories, the $\mathcal{N}=2$ central charge is a linear combination of k integers.

Consider firstly the Coulomb branch of Model B, as described in the appendix. The possible BPS states of the theory, which lie in short representations of the supersymmetry algebra, are associated to (suitably ordered) products of chiral operators,

$$X(n_i) = \prod_{i=1}^N \widehat{\Phi}_i^{n_i} \tag{5.1}$$

i.e. $n_i \geq 0$ electrons of the i^{th} type. However, in three-dimensional gauge theories with massless gauge bosons, the $1/r$ fall-off of electric fields ensures that any state charged under a local current has logarithmically divergent mass. Thus, on the Coulomb branch, all finite mass states in the theory are associated to gauge invariant

[2]It is curious to note that the second terms in (4.4) and (4.8), which are both vanishing in the two-dimensional continuum limit, differs only by a factor of 2. It would be interesting to understand why the behaviour is similar and why they differ by the factor of 2.

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operators. The operator $X(n_i)$ is gauge invariant if and only if the positive integers n_i satisfy,

$$\sum_{i=1}^{N} n_i \widehat{Q}_i^p = 0 \qquad \forall\, p = 1, \ldots, N-k\,. \tag{5.2}$$

The solutions to this equation, if they exist, are given by $n_i = \sum_{a=1}^{k} p_a Q_i^a$ for some integers $p_1, \ldots, p_k$. The integers p_a can be identified as the Noether charge of $X(n_i)$ associated with the $\mathrm{U}(1)^k$ global symmetry group.[3] Thus, if there exists a k-vector charge p_a such that the resulting n_i are all non-negative, then there exists a corresponding BPS bound state of $\sum_{i=1}^{N} n_i$ logarithmically confined electrons. In this case the mass of the state is determined by the central charge, which is not renormalized, and is given by,

$$M(n_i) = \sum_{i=1}^{N} n_i \widehat{m}_i = \sum_{a=1}^{k} \left(\sum_{i=1}^{N} Q_i^a \widehat{m}_i \right) p_a\,. \tag{5.3}$$

The gauge invariance of this state ensures that the mass is independent of $\hat{\phi}_p$, the position on the Coulomb branch. Indeed the sum of the effective mass $\widehat{M}_i = \widehat{Q}_i^p \hat{\phi}_p + \widehat{m}_i$ of the constituents is the same as the central charge (5.3), $\sum_{i=1}^{N} n_i \widehat{M}_i = \sum_{i=1}^{N} n_i \widehat{m}_i = M(n_i)$.

For a charge p_a such that some of the resulting n_i are negative, one cannot find a gauge invariant chiral operator as a combination of positive powers of $\widehat{\Phi}_i$'s. Instead, there exists an operator that involves both chiral and anti-chiral superfields;

$$\tilde{X}(n_i) = \prod_{i \in I} \widehat{\Phi}_i^{n_i} \prod_{j \notin I} (\widehat{\Phi}_j^\dagger)^{-n_j}\,, \tag{5.4}$$

where

$$I := \{\, i\,;\, n_i \geq 0\}\,. \tag{5.5}$$

This is gauge invariant (in the Wess-Zumino gauge) and has the right charge p_a under $\mathrm{U}(1)^k$ global symmetry group. The mass of the state associated to $\tilde{X}(n_i)$ is subject to renormalization. In fact, the sum of the effective masses $\sum_{i=1}^{N} |n_i| \widehat{M}_i$ depends on the values of $\widehat{\phi}_p$ and is in general larger than the central charge (5.3). Nevertheless, if we restrict to the locus on the Coulomb branch,

$$\sum_{p=1}^{N-k} Q_i^p \hat{\phi}_p + \widehat{m}_i = 0 \qquad \forall\, i \notin I\,, \tag{5.6}$$

then the effective mass of the electrons $\widehat{\Phi}_i$ $(i \notin I)$ is vanishing and the sum $\sum_{i=1}^{N} |n_i| \times \widehat{M}_i$ becomes $\sum_{i=1}^{N} n_i \widehat{M}_i = M(n_i)$ and classically saturates the BPS bound. However, there is still room for renormalization and the state associated to $\tilde{X}(n_i)$ does not lie in a short multiplet.

[3] The global symmetry $\mathrm{U}(1)^k$ acts on $\widehat{\Phi}_i$ with charge the $\widehat{R}_{ia}$ complementary to $\widehat{Q}_i^p$. If $\widehat{R}_{ia}$ are chosen to obey $\sum_{i=1}^{N} Q_i^a \widehat{R}_{ib} = \delta_b^a$, the charge of $X(n_i)$ is $\sum_{i=1}^{N} n_i \widehat{R}_{ia} = \sum_{i,b} p_b Q_i^b \widehat{R}_{ia} = p_a$.

Let us now turn to the Higgs branch of Model A where the particle states are Nielsen-Olesen vortices. We consider a vortex arising from non-trivial winding of the U(1) subgroup of the gauge group $U(1)^k$, defined by $e^{i\theta} \hookrightarrow (e^{ip_a\theta})$, under which the chiral multiplet Φ_i has charge $n_i = \sum_{a=1}^{k} p_a Q_i^a$. Note that we have employed the same notation, p_a and n_i, as used in the discussion of the Coulomb branch and, indeed, these quantities will be identified with each other under mirror symmetry. Let us examine under what circumstances the vortices saturates the classical BPS bound. Specifically, consider a vortex with unit flux $\frac{-1}{2\pi}\int dv = 1$. While such vortices exist at all points of the Higgs branch, they saturate the classical BPS bound only at specific points. To see this, we need the result that the Bogomoln'yi equations can only be satisified if [17]

$$\Phi_i = 0 \qquad \forall\ i \notin I\,. \tag{5.7}$$

This requirement is equivalent to the statement that a line bundle of negative degree has no non-zero holomorphic section. Notice that this restriction to loci of the Higgs branch is mapped under mirror symmetry to (5.6).

Now let us ask under what circumstance the vortices lie in short representations of the supersymmetry algebra. To see this we must quantize the zero-modes of the vortex. The vortex system has $\sum_{i\in I} n_i$ bosonic zero modes [18].[4] There are also fermionic zero modes. In three-dimensions each chiral multiplet, Φ_i, contains a single Dirac fermion ψ_i. For $i \in I$, each ψ_i has n_i zero modes while the conjugate spinor $\bar{\psi}_i$ has none.[5] These are paired by the unbroken supersymmetry of the vortex with the bosonic zero modes. However, fermionic zero modes arise also from Φ_i with $i \notin I$. In this case ψ_i has no zero modes, while $\bar{\psi}_i$ has $-n_i$, which are not paired with bosonic zero modes. Thus the total number of fermionic zero modes is given by $\sum_{i=1}^{N} |n_i|$. This mismatch of bosonic and fermionic zero modes reflects the fact that the vortices lie in long multiplets unless $n_i \geq 0$ for all i, in agreement with the situation on the Coulomb branch. Note that when this occurs, the mass of the BPS vortex is given by

$$M(n_i) = \sum_{i=1}^{k} p_a \zeta_a \tag{5.8}$$

which agrees with (5.3) under the identification of the mirror map (note that we have set $m_i = 0$ for Model A).

We note that the number $\sum_{i\in I} n_i$ of bosonic zero modes of a Model A vortex agrees with the number of holomorphic constituent components of the logarithmically confined states on the Model B Coulomb branch. This might suggest that the electron $\widehat{\Phi}_i$ of Model B can be interpreted as the fractional vortex which has winding number

[4]To see this, one must adapt the results of [18] to *based* vortices which asymptote to the vacuum at a given point on the $\mathbf{P}^1$ worldsheet.

[5]Upon dimensional reduction to two dimensions only, say, left moving spinors have zero modes while right-movers have none.

one in the Φ_i component. This picture is in fact consistent with the generation of the superpotential (3.27) in the compactified theory, in comparison with the two-dimensional derivation [1]. We have seen that each term $\mathrm{e}^{-\widehat{Q}_i^p\Theta_p - r_i}$ is generated by the loop of $\widehat{\Phi}_i$ Kaluza-Klein modes. On the other hand, in two-dimensions, the same term is generated by a one Φ_i vortex-instanton in the theory with an extended gauge symmetry [1]. It would also be interesting to analyze the fractional instanton effect, directly in the theory without extended gauge symmetry.

6. Concluding remarks

In this section we would like to briefly summarize the calculations performed in this paper, and suggest some avenues for further research.

The primary accomplishment was to compactify 2+1-dimensional abelian-Chern-Simons mirror pairs with four supercharges on a circle of radius R. We showed that in order to arrive at a non-trivial interacting theory in the continuum limit $R \to 0$, both the parameters and the fields of the three-dimensional theory must scale with R. When this scaling is performed, the theory on the Higgs branch, denoted as Model A in the text, reduces to a $1+1$-dimensional non-linear sigma model with a toric target space. The tower of KK modes play no role for this theory, decoupling as $R \to 0$. In contrast, for the theory on the Coulomb branch, denoted as Model B, the necessary scalings ensure that the KK modes do not decouple. Rather, each contributes to the superpotential and the total may be resummed, resulting in the exact superpotential (3.21). In the limit $R \to 0$, this reduces to the Toda-like LG superpotential, which was shown in [1] to be dual to the toric sigma-model.

It is noteworthy that the resummation of loop effects of the infinite tower of KK modes captures non-perturbative effects in 1+1 dimensions. This phenomenon is not new: it has occured previously in compactification from 3+1 to 2+1 dimensions [19], as well as in compactification from $4+1$ to $3+1$ dimensions [20].

As mentioned in the introduction, the mirror transformation in three dimensions may be thought of as a generalization of scalar-vector duality. In particular, the variable parameterizing the Higgs branch of Model A is mapped to the dual photon of Model B. However, when considering the compactification of Model B, we worked not with the dual photon, but rather with the original gauge field, in the guise of a Wilson line, and the resulting LG theory is a description in terms of these variables. As discussed in section 2.2, the scalar-vector duality of free-abelian theories in three-dimension reduces upon dimensional reduction to scalar-scalar duality in two-dimensions. One of the main messages of this paper is that, at least for the class of models considered, this correspondence continues to hold even in the interacting case.

In section 4, we analyzed in more detail the dependance of the compactification on coupling constants of Model A and Model B and, in particular, on the dimensionless ratios, $\gamma = e_{3\mathrm{d}}^2 R$ and $\widehat{\gamma} = \widehat{e}_{3d}^2 R$. We noted that the derivation given in Section

3 was valid only in the regime $\widehat{\gamma} \ll 1$, while the use of three-dimensional mirror symmetry as an infra-red duality would require $\widehat{\gamma} \gg 1$. We interpreted the fact that we obtained the correct answer as evidence for the lack of a phase transition as $\widehat{\gamma}$ is varied. Moreover, we repeated our discussion for the all-scale three-dimensional mirror pairs conjectured in [7]. To extend mirror symmetry beyond the infra-red limit requires a modification of the theory on the Higgs branch; this was denoted Model A′ in the text. As a test of the conjecture that these theories are indeed mirror on all length scales, we examined the Kahler potential, a quantity not protected by supersymmetry. We found that a finite Kahler potential for the LG theory in $1+1$ dimensions corresponds to a squashing of the metric of the toric sigma-model, in agreement with expectations [8].

6.1 Future directions

There remain several open questions, some of which we list here. Firstly, we considered only abelian gauge theories in this paper. In $1+1$ dimensions, the derivation of mirror symmetry given in [1] does not straightforwardly apply to non-abelian theories. Moreover, the dual of non-linear sigma models arising from non-abelian gauge theories (such as Grassmannians [21]) are unknown, although there are indications of the existence of LG type mirrors [22]. In contrast, there exist many three-dimensional non-abelian mirror pairs. One may therefore use the techniques presented here in order to find new non-abelian mirror pairs in two dimensions.

Secondly, an extremely interesting class of mirror pairs in $1+1$ dimensions are the $(2,2)$ superconformal theories with compact Calabi-Yau target spaces. For reviews and references see [23, 1, 24]. It would be interesting if this could also be related three-dimensional theories.

Finally, as shown in [5] (and also in appendix A), our three-dimensional $\mathcal{N}=2$ mirror theories can be obtained from the RG flow of $\mathcal{N}=4$ mirror theories perturbed by an operator which partially breaks supersymmetry. In other words, the route we have taken to arrive at $1+1$ dimensional mirrors is: 3d $\mathcal{N}=4 \Rightarrow$3d $\mathcal{N}=2 \Rightarrow$ 2d $(2,2)$. It is natural to ask[6] if one may exchange the order of partial supersymmetry breaking and compactification: 3d $\mathcal{N}=4 \Rightarrow$ 2d $(4,4) \Rightarrow$ 2d $(2,2)$. Along this route, one first compactifies the $\mathcal{N}=4$ mirror pairs to obtain duality between two-dimensional $(4,4)$ theories [9], before deforming the resulting pairs by an operator that breaks $(4,4)$ supersymmetry to $(2,2)$.

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A. Regularization and the mirror map

In this appendix, we describe the regularization scheme of Model A and B which has been used in the text so far. In particular, we derive the mirror map (3.3) and also explain why the second term in (3.19)

$$-\sum_{i=1}^{N} \widehat{Q}_i^p \log\left[\frac{1}{2\pi R}\prod_{n=1}^{\infty}\left(\frac{n^2}{R^2}\right)\right] \tag{A.1}$$

can be set equal to zero. We will also show that the final result (3.27) is essentially independent of the choice of the regularization scheme.

A.1 Deformation of finite $\mathcal{N}=4$ mirror pairs

Model A is not finite unless $\sum_{i=1}^{N} Q_i^a = 0$ for all a (and similarly for Model B); The one-point function of the auxiliary field

$$\langle D_a\rangle \propto \sum_{i=1}^{N} Q_i^a \int \frac{\mathrm{d}^3k}{k^2+\cdots}\,, \tag{A.2}$$

is linearly divergent. Thus, both Model A and Model B have to be regularized and the FI parameters must be renormalized. We will show that the finite $\mathcal{N}=4$ mirror pairs can be used as the cut-off theories.

We consider the following $\mathcal{N}=4$ pair from [12].

Model A$_{(4)}$: $\mathrm{U}(1)^k$ gauge theory of vector multiplets (V_a,Ψ_a) with N hypermultiplets $H_i=(\Phi_i,\Phi_i^\vee)$ of charge Q_i^a. Only the real FI parameters $\zeta_{(4)}^a$ and the real masses $m_i^{(4)}$ for H_i are turned on.

Model B$_{(4)}$: $\mathrm{U}(1)^{N-k}$ gauge theory of $(\widehat{V}_p,\widehat{\Psi}_p)$ with N hypermultiplets $\widehat{H}_i=(\widehat{\Phi}_i,\widehat{\Phi}_i^\vee)$ of charge $\widehat{Q}_i^p$. The real FI parameters $\widehat{\zeta}_{(4)}^p$ and the real masses $\widehat{m}_i^{(4)}$ for $\widehat{H}_i$ are turned on, but the complex FI/masses are turned off.

The parameters are related by the $\mathcal{N}=4$ mirror map

$$\zeta_{(4)}^a=\sum_{i=1}^{N} Q_i^a\widehat{m}_i^{(4)} \qquad , -\sum_{i=1}^{N}\widehat{Q}_i^p m_i^{(4)}=\widehat{\zeta}_{(4)}^p\,. \tag{A.3}$$

As in [5], we give a background value X to the scalar component of the vector multiplet for a U(1) subgroup of the R-symmetry. This gives a mass $2X$ to Ψ_a and $\widehat{\Psi}_p$, and the masses of the hypermultiplet fields are changed to

$$\begin{aligned} m^2(\Phi_i) &= |Q_i^a\Psi_a|^2 + (Q_i^a\phi_a + m_i^{(4)} + X)^2\,, \\ m^2(\Phi_i^\vee) &= |Q_i^a\Psi_a|^2 + (-Q_i^a\phi_a - m_i^{(4)} + X)^2\,, \\ m^2(\widehat{\Phi}_i) &= |\widehat{Q}_i^p\widehat{\Psi}_p|^2 + (\widehat{Q}_i^p\widehat{\phi}_p + \widehat{m}_i^{(4)} - X)^2\,, \\ m^2(\widehat{\Phi}_i^\vee) &= |\widehat{Q}_i^p\widehat{\Psi}_p|^2 + (-\widehat{Q}_i^p\widehat{\phi}_p - \widehat{m}_i^{(4)} - X)^2\,. \end{aligned} \tag{A.4}$$

We will send X to infinity, keeeping fixed the following mass parameters

$$m_i = m_i^{(4)} + X\,, \qquad \widehat{m}_i = \widehat{m}_i^{(4)} - X\,. \tag{A.5}$$

Then, Φ_i, $\widehat{\Phi}_i$ have finite masses but Ψ_a, $\widehat{\Psi}_p$, $\Phi_i^\vee$ and $\widehat{\Phi}_i^\vee$ have masses that diverges as $\sim 2X$. Thus, it is appropriate to integrate out these heavy fields. Integration of the neutral fields Ψ_a, $\widehat{\Psi}_p$ simply sets these fields to zero. Integration of the charged fields generates Chern-Simons and FI terms for the $\mathcal{N}=2$ vector multiplets. To find the precise form, we compactify the theory on the circle of radius R where the three-dimensional theory is recovered by taking the decompactification limit $R\to\infty$. This is simply for a technical reason; one can use the machinery of computation developed in section 3.3.

As in section 3.3, the generated superpotential $\Delta\widetilde{W}$ in Model $\mathrm{A}_{(4)}$ is

$$\frac{\partial\Delta\widetilde{W}}{\partial\Sigma_a} = \sum_{i=1}^{N} Q_i^a \log(\mathrm{e}^{\pi R\Sigma_i} - \mathrm{e}^{-\pi R\Sigma_i}) + \sum_{i=1}^{N} Q_i^a \log Z\,, \tag{A.6}$$

where $\Sigma_i = -Q_i^a\Sigma_a - m_i + 2X$ and Z is the infinite product

$$Z = \frac{1}{2\pi R}\prod_{n=1}^{\infty}\frac{n^2}{R^2}, \tag{A.7}$$

that appears also in (3.19). Since Σ_i is large in the limit $X\to+\infty$, $\mathrm{e}^{\pi R\Sigma_i}$ is dominant in the log of the first term in (A.6), compared to $\mathrm{e}^{-\pi R\Sigma_i}$ that vanishes in the decompactification limit $R\to\infty$. Thus, (A.6) is essentially $\sum_{i=1}^N Q_i^a(\pi R\Sigma_i + \log Z)$. By integration, we find

$$\Delta\widetilde{W} = -\frac{\pi R}{2}\sum_{i,a,b} Q_i^a Q_i^b \Sigma_a\Sigma_b - 2\pi R\sum_a \left(\frac{1}{2}\sum_i Q_i^a m_i - \sum_i Q_i^a (X + \frac{1}{2\pi R}\log Z)\right)\Sigma_a\,. \tag{A.8}$$

Comparing with (2.5) and (2.7), we find that the CS and FI couplings are given by

$$k^{ab} = \frac{1}{2}\sum_{i=1}^{N} Q_i^a Q_i^b\,, \tag{A.9}$$

$$\zeta^a = \zeta_{(4)}^a + \frac{1}{2}\sum_{i=1}^{N} Q_i^a m_i - \sum_{i=1}^{N} Q_i^a\left(X + \frac{1}{2\pi R}\log Z\right). \tag{A.10}$$

The same procedure gives the CS and FI coupling of Model B;

$$\widehat{k}^{pq} = -\frac{1}{2}\sum_{i=1}^{N}\widehat{Q}_i^p\widehat{Q}_i^q\,,$$

$$\widehat{\zeta}^p = \widehat{\zeta}^p_{(4)} - \frac{1}{2}\sum_{i=1}^{N}\widehat{Q}_i^p\widehat{m}_i - \sum_{i=1}^{N}\widehat{Q}_i^p\left(X + \frac{1}{2\pi R}\log Z\right). \tag{A.11}$$

The change in the sign compared to (A.9)–(A.10) is because $\widehat{\Sigma}_i = -\widehat{Q}_i^p\widehat{\Sigma}_p - \widehat{m}_i - 2X$ have large *negative* values.

A.2 Zeta function regularization

We notice that the last terms in (A.10) and (A.11) are divergent, which is related to the divergence (A.2). At this point, we treat the infinite product (A.7) by zeta function regularization. Defining $\zeta_R(s) = \sum_{n=1}^{\infty}(n/R)^{-s} = R^s\zeta(s)$ and noting that $\zeta(0) = -\frac{1}{2}, \zeta'(0) = -\frac{1}{2}\log(2\pi)$, we find

$$Z = \frac{1}{2\pi R}\mathrm{e}^{-2\zeta_R'(0)} = \frac{1}{2\pi R}\mathrm{e}^{-2\zeta'(0)-2\zeta(0)\log R} = 1\,. \tag{A.12}$$

Then, the $\log Z$ terms in (A.10) and (A.11) vanish. Using the $\mathcal{N}=4$ mirror map (A.3) and (A.5), we find that the X linear terms also cancel out;

$$\zeta^a = \sum_{i=1}^{N}Q_i^a\widehat{m}_i + \frac{1}{2}\sum_{i=1}^{N}Q_i^a m_i\,, \tag{A.13}$$

$$\widehat{\zeta}^p = -\sum_{i=1}^{N}\widehat{Q}_i^p m_i - \frac{1}{2}\sum_{i=1}^{N}\widehat{Q}_i^p\widehat{m}_i\,, \tag{A.14}$$

In this way, we recover the $\mathcal{N}=2$ mirror map (3.3). We have seen that this mirror map is based on the zeta function regularization (A.12). It is also clear that the second term of (3.19) should be set equal to zero under this regularization scheme.

A.3 Other regularization schemes

We could have chosen another regularization scheme where $\log Z$ does indeed diverge. First, it is easy to see that the $\log Z$ term in $\widehat{\zeta}^p$ of (A.11) cancels the second term of (3.19) when we add $\widetilde{W}_{FI} + \widetilde{W}_{CS}$ to $\Delta\widetilde{W}$. What about the $\log Z$ term in (A.10)?

To examine this, let us look at the effective FI coupling on the Higgs branch of Model A (where we set $m_i = 0$). The third term in (A.10) can be considered as an expression of the one-loop integral of $\Phi_i^\vee$

$$-\sum_{i=1}^{N}Q_i^a\left(X + \frac{\log Z}{2\pi R}\right) = \sum_{i=1}^{N}Q_i^a\frac{1}{2\pi R}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^2k}{2\pi}\frac{1}{k^2+\frac{n^2}{R^2}+(2X)^2}$$
$$= \sum_{i=1}^{N}Q_i^a\int\frac{\mathrm{d}^3k}{(2\pi)^2}\frac{1}{k^2+(2X)^2}\,. \tag{A.15}$$

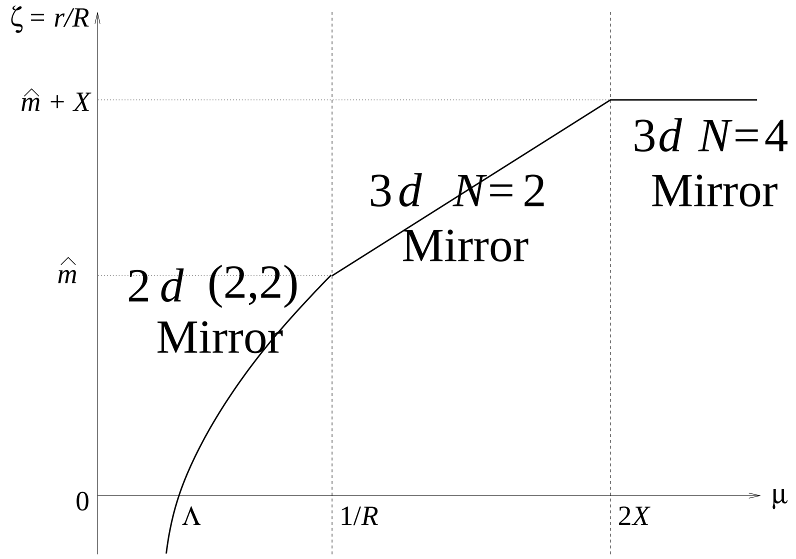

Figure 1: Mirror symmetry at various scales.

The effective FI coupling at energy μ is obtained by restricting this integral in the range $|k| \geq \mu$ and adding the contribution from the Φ_i loop integral as well;

$$\begin{aligned} \zeta^a(\mu) &= \zeta^a_{(4)} + \sum_{i=1}^{N} Q_i^a \left[\int_{|k|\geq\mu} \frac{\mathrm{d}^3k}{(2\pi)^2} \frac{1}{k^2+(2X)^2} - \int_{|k|\geq\mu} \frac{\mathrm{d}^3k}{(2\pi)^2}\frac{1}{k^2} \right] \\ &= \begin{cases} \zeta^a_{(4)} & \mu \gg 2X, \\ \zeta^a_{(4)} + \sum_{i=1}^N Q_i^a(\mu/\pi - X) & \mu \ll 2X\,. \end{cases} \end{aligned} \tag{A.16}$$

As noted before, $\zeta^a_{(4)}$ is chosen to be $\zeta^a_{(4)} = \sum_{i=1}^N Q_i^a(\widehat{m}_i + X)$ and it cancels the X linear term, yielding

$$\zeta^a(\mu) = \sum_{i=1}^{N} Q_i^a \widehat{m}_i + \sum_{1=1}^{N} Q_i^a \mu/\pi\,, \qquad \mu \ll 2X\,. \tag{A.17}$$

We find that $2X$ serves as the cut-off of Model A and the cut-off theory is provided by the finite Model $\mathrm{A}_{(4)}$.

Upon compactification, the 2d sigma model is obtained by renomalizing the FI parameter according to the RG matching equation

$$\zeta^a(1/2\pi R) = \frac{1}{2\pi R}\left(\sum_{i=1}^{N} Q_i^a \log(1/2\pi R\Lambda) + r^a \right). \tag{A.18}$$

Using (A.17), this dictates the R dependence of $\widehat{m}_i$, as in (3.12). The presence of the term $\sum_{i=1}^N Q_i^a\mu/\pi$ in $\zeta^a(\mu)$ only yields a shift of r_i in (3.12) by $-1/\pi$. This simply gives a finite rescaling of the superpotential $\widetilde{W}$ in the final result (3.27).

The results of this appendix, and in particular the running of FI parameter, are summarized in figure 1.

B. The Coulomb branch in three dimensions

In this appendix we briefly review the results of [6], explaining how the Coulomb branch of Model B comes about. We further determine the Kahler class of the moduli space and show that it recieves no quantum corrections beyond one-loop.

To determine the Coulomb branch of Model B, we begin with an examination of the classical potential energy,

$$V=\sum_{p=1}^{N-k}\widehat{e}_p^2\left(\sum_{i=1}^{N}\widehat{Q}_i^p|\widehat{\Phi}_i|^2-\sum_{q=1}^{N-k}\widehat{k}^{pq}\hat{\phi}_q-\widehat{\zeta}^p\right)^2+\sum_{i=1}^{N}\widehat{M}_i^2|\widehat{\Phi}_i|^2\,. \tag{B.1}$$

Recall that $\widehat{\phi}_p$ are the real scalars of the vector multiplets while $\widehat{\Phi}_i$ are the complex scalars of the chiral multiplets. The latter have real masses $\widehat{M}_i$ given by,

$$\widehat{M}_i=\sum_{p=1}^{N-k}\widehat{Q}_i^p\widehat{\phi}_p+\widehat{m}_i\,. \tag{B.2}$$

Note in particular that the D-term (the first term in (B.1)) includes the supersymmetric completion of the Chern-Simons coupling [25] which, in our case, has coefficient

$$\widehat{k}^{pq}=-\tfrac{1}{2}\sum_{i=1}^{N}\widehat{Q}_i^p\widehat{Q}_i^q\,. \tag{B.3}$$

At first glance, the appearance of this term in the potential seems to rule out the possibility of a Coulomb branch. Indeed, setting the chiral multiplets to zero, a non-zero expectation value for $\widehat{\phi}_p$ results in a non-zero energy,

$$V=\sum_{p=1}^{N-k}\widehat{e}_p^2\left(\sum_{q=1}^{N-k}\widehat{k}^{pq}\hat{\phi}_q+\widehat{\zeta}^p\right)^2\,. \tag{B.4}$$

The fact that the FI parameters lift the Coulomb branch is well-known, while the fact that the supersymmetric completion of the Chern-Simons couplings perform a similar feat can easily be anticipated; the scalars $\widehat{\phi}_p$ are the superpartners of the gauge fields, and the latter acquire gauge invariant masses from the Chern-Simons couplings.

While there is no classical Coulomb branch, the siutation is quite different quantum mechanically, for both the FI and Chern-Simons parameters are renormalized upon integrating out the chiral multiplets. While perturbation theory is valid only in the regime $\widehat{M}_i\gg\widehat{e}_p^2$ for all i,p, the topological nature of both couplings[7] ensures

[7]Note that the FI parameter may be thought of as a mixed Chern-Simons coupling between global and local currents and therefore enjoys the same one-loop non-renormalization theorem.

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that the one-loop result is exact [26], and given by,

$$\widehat{k}^{pq}_{\rm eff} = -\frac{1}{2}\sum_{i=1}^{N}\widehat{Q}^p_i\widehat{Q}^q_i + \frac{1}{2}\sum_{i=1}^{N}\widehat{Q}^p_i\widehat{Q}^q_i\,{\rm sign}\,(\widehat{M}_i)$$

$$\widehat{\zeta}^p_{\rm eff} = \widehat{\zeta}^p + \frac{1}{2}\sum_{i=1}^{N}\widehat{Q}^p_i\widehat{m}_i\,{\rm sign}\,(\widehat{M}_i)\,.$$

The effective potential is then given by (B.4) with the parameters replaced by their quantum corrected versions. We see that the Chern-Simons coupling vanishes only in the regime,

$$\widehat{M}_i \geq 0 \tag{B.5}$$

while the effective FI parameter also vanishes in this regime if the bare FI parameter is given by $\widehat{\zeta}^p = \frac{1}{2}\sum_i \widehat{Q}^p_i\widehat{m}_i$, in agreement with the mirror map (3.3).

The N inequalities (B.5) define a region, $\Delta \subset \mathbf{R}^{N-k}$, parameterized by $\widehat{\phi}_p$. Δ may or may not be compact depending on the charges $\widehat{Q}^p_i$. This defines "half" of the Coulomb branch. The other half comes from the dual photons, $\widehat{\varphi}^p$. These parameterize a torus, $\mathbf{T}^{N-k}$, which is fibered over Δ. Moreover, at the boundaries of Δ, given by the equations $\widehat{M}_i = 0$, certain cycles of $\mathbf{T}^{N-k}$ degenerate. To see this, note that whenever $\widehat{M}_i = 0$, we have integrated out a chiral multiplet of vanishing mass, and we therefore expect the linear combination of gauge fields under which $\widehat{\Phi}_i$ is charged to have a (possibly coordinate) singularity in the low-energy effective action. This in turn implies the degeneration of the cycle $\widehat{\varphi}^p \propto \widehat{Q}^p_i$. One can also argue that this cycle vanishes using the symmetries of the dual photon [4, 6].

Thus we arrive at the Coulomb branch of Model B; a torus $\mathbf{T}^{N-k}$ fibered over a region $\Delta \subset \mathbf{R}^{N-k}$. In [6], it was shown that the one-loop Coulomb branch coincides as a toric variety with the Higgs branch of Model A. Moreover, the two moduli spaces have the same isometries, with the same fixed points, which is a statement that holds non-perturbatively. We now determine the exact Kahler class of the Coulomb branch and show that it is not renormalized by quantum effects. To do this, we examine the expression for the supersymmetric low-energy effective action which, up to two derivatives, is given by

$$\mathcal{L} = \int {\rm d}^4\theta f(\widehat{G}_p)\,. \tag{B.6}$$

Here $\widehat{G}_p$ is the linear superfields which contain the scalars $\widehat{\phi}_p$ and the field strengths $\widehat{v}_{p\mu\nu}$. A linear superfield G has the following expansion (we set the fermionic components zero)

$$\begin{aligned} G = \phi &+ \frac{1}{2}\theta^+\overline{\theta}^-(D - iv_{01}) + \frac{1}{2}\theta^-\overline{\theta}^+(D + iv_{01}) \\ &+ \frac{1}{2}\theta^+\overline{\theta}^+(-v_{02} - v_{12}) + \frac{1}{2}\theta^-\overline{\theta}^-(v_{02} - v_{12}) + \frac{1}{4}\theta^+\theta^-\overline{\theta}^-\overline{\theta}^+(\partial_0^2 - \partial_1^2 - \partial_2^2)\phi\,. \end{aligned}$$

The crucial point here is that the fieldstrength $v_{\mu\nu}$ appears only in the $\theta\overline{\theta}$ terms whereas ϕ appears in the θ^0 and $\theta^2\overline{\theta}^2$ terms. The bosonic part of the Lagrangian therefore reads

$$\mathcal{L}_b = g^{pq}\left(\partial_\mu\widehat{\phi}_p\partial^\mu\widehat{\phi}_q + \frac{1}{2}\widehat{v}_{p\mu\nu}\widehat{v}_q^{\mu\nu}\right), \tag{B.7}$$

where $g^{pq} := \partial^2 f/\partial\widehat{\phi}_p\partial\widehat{\phi}_q$. After dualization we obtain

$$\mathcal{L}_b = g^{pq}\partial_\mu\widehat{\phi}_p\partial^\mu\widehat{\phi}_q + g_{pq}\partial_\mu\widehat{\varphi}^p\partial^\mu\widehat{\varphi}^q \tag{B.8}$$

where g_{pq} is the inverse matrix of g^{pq}. In particular, there are no cross-terms between $\widehat{\phi}_p$ and $\widehat{\varphi}^p$. This is unlike the effective action for $\mathcal{N}=4$ vector multiplet which also contains an $\mathcal{N}=2$ chiral multiplet in a superpotential and does indeed lead to such cross-terms [27].

The metric on the Coulomb branch (B.8) can be written as $\mathrm{d}s^2 = g_{pq}\mathrm{d}z^p\mathrm{d}\overline{z}^q$ where

$$z^p = \frac{\partial f}{\partial\widehat{\phi}_p} + i\widehat{\varphi}^p\,, \tag{B.9}$$

and this implies that z^p are complex coordinates on the Coulomb branch. (This can also be proved by dualization on the superspace [28]. See also [29, section 2.3].) Thus, the Kahler form is given by

$$\begin{aligned}\Omega &= \frac{i}{2}g_{pq}\mathrm{d}z^p\wedge\mathrm{d}\overline{z}^q = g_{pq}\mathrm{d}\left(\frac{\partial f}{\partial\widehat{\phi}_p}\right)\wedge\mathrm{d}\widehat{\varphi}^q\\ &= \mathrm{d}\widehat{\phi}_p\wedge\mathrm{d}\widehat{\varphi}^p.\end{aligned} \tag{B.10}$$

The Kahler class is determined by measuring the area of 2-cycles using Ω. A typical 2-cycle is the circle fibration over a segment between two vertices of Δ, where the circle shrinks to zero size at the two ends. Its Ω area is just 2π times the length of the segment, which is determined exactly at the one-loop level. Thus, the Kahler class is independent of the details of the function f and, in particular, is not corrected from the one-loop result by further quantum effects. Moreover, Ω coincides with the Kahler form of the Higgs branch of Model A.

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